ORBIFOLD CONSTRUCTIONS OF $K3$: A LINK BETWEEN CONFORMAL FIELD THEORY AND GEOMETRY

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Abstract. We discuss geometric aspects of orbifold conformal field theories in the moduli space of $N = (4,4)$ superconformal field theories with central charge $c = 6$. Part of this note consists of a summary of our earlier results on the location of these theories within the moduli space [?, ?] and the action of a specific version of mirror symmetry on them [?]. We argue that these results allow for a direct translation from geometric to conformal field theoretic data. Additionally, this work contains a detailed discussion of an example which allows the application of various versions of mirror symmetry on $K3$. We show that all of them agree in that point of the moduli space.

Introduction

This note is intended to make a contribution to the understanding of links between algebraic geometry and theoretical physics, with an emphasis on geometric aspects of conformal field theory.

From the set up of string theory, a connection to geometry is more or less obvious, but in general it seems to be hard to formulate it in precise mathematical terms. Nevertheless, many aspects of string theory have (had) a strong influence on mathematics, among them ORBIFOLD STRING THEORY and MIRROR SYMMETRY, both of which are leitmotifs for the present note. String theory at small coupling is described by a superconformal field theory (SCFT) on the world sheet. Therefore, a connection between geometry and SCFT is expected, too. Since SCFTs are well-defined mathematical objects in their own right, an investigation of direct links to geometry offers a mathematically safe basis which we are using in this work.

We focus on a special type of SCFTs which is simple enough to carry out a sound analysis but also provides enough non-trivial structure to find interesting links to geometry: We investigate aspects of the moduli space of those SCFTs with central charge $c = 6$ whose Hilbert space is a representation of a specific $N = (4,4)$ superconformal algebra $\mathcal{A}$. Namely, $\mathcal{A}$ contains an affine $su(2)_l \oplus su(2)_r$ Kac-Moody algebra at level 1 [?], such that all left and right charges with respect to a Cartan subalgebra of $su(2)_l \oplus su(2)_r$ (i.e. all doubled spins) are integral. We are working with a partial completion $\mathcal{M}$ of those two components of this moduli space that are relevant in string theory, which for simplicity we call the MODULI SPACE OF $N = (4,4)$ SCFTS WITH CENTRAL CHARGE $c = 6$.

One reason to make such concessions is the fact that the space $\mathcal{M}$ is known explicitly [?, ?, ?, ?, ?], and its very description already allows to draw links between geometric and superconformal field theoretic data: Its two connected components $\mathcal{M}^{tori}$, $\mathcal{M}^{K3}$ are naturally interpreted as extensions of the geometric moduli spaces

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of Einstein metrics on a complex two torus and a $K3$ surface, respectively. On the other hand, statements about orbifold CFTs and mirror symmetry for theories in $\mathcal{M}$ are non-trivial already.

We will begin with a summary of what is known about the moduli space $\mathcal{M}$. It in particular includes the precise description of the location of orbifold CFTs of toroidal theories within $\mathcal{M}$ that was obtained in [?, ?]. This requires the determination of certain B-field values which at first sight might appear to be a physicist's invention that is hard to assign an intrinsic geometric meaning to. However, we will argue that orbifold CFTs on $K3$ allow for an explicit geometric explanation of the B-field by the use of the classical McKay correspondence. This viewpoint is somewhat complementary to the one taken in [?, ?, ?] but proves to be particularly useful in the present context. A second topic of this note is the discussion of mirror symmetry on $\mathbb{Z}_N$ orbifold CFTs on $K3$, which summarizes the results of [?]. We use a geometric approach by fiberwise T-duality on a torus fibration of the manifold under discussion, which goes back to ideas by Vafa and Witten [?]. It allows us to determine the mirror map for $\mathbb{Z}_N$ orbifold CFTs on $K3$ explicitly. Since both the geometric and the conformal field theoretic approaches are totally under control, it also provides an explicit translation between geometric and conformal field theoretic data. The third and last part of the present work is devoted to a detailed discussion of mirror symmetry for a particular SCFT in $\mathcal{M}$. This theory allows a comparison of our approach to mirror symmetry with two other versions that have been successfully applied to $K3$ before [?, ?, ?, ?]. We show that with an emendation due to Rohsiepe [?, ?] all these approaches are compatible for our example.

This work is organized as follows: In Sect. 1 we give a brief description of the moduli space $\mathcal{M}$ as algebraic space. Section 2 explains the location of orbifold CFTs of toroidal theories within $\mathcal{M}$ and in particular provides a solution to the "B-field problem" in terms of geometric structures. In Sect. 3 we give a somewhat superficial introduction to those aspects of mirror symmetry that are relevant for our discussion of $\mathcal{M}$. The version of mirror symmetry which is induced by fiberwise T-duality on a specific elliptic fibration of a $\mathbb{Z}_N$ orbifold limit of $K3$ gives another direct link between geometry and SCFT. This is explained in Sect. 4. Section 5 contains the discussion of a particular theory in $\mathcal{M}$. We compare three versions of mirror symmetry on $K3$ and show that they agree for this theory. We end with a summary and discussion in Sect. 6.

The aim of this note is to give a digestible overview on the subject and to explain the above mentioned example. In particular, proofs that are already written up elsewhere (see [?, ?, ?, ?]) are omitted.

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1. The moduli space $\mathcal{M}$ of $N=(4,4)$ SCFTs with $c=6$

Let us briefly describe the structure of the moduli space $\mathcal{M}$ [?, ?, ?, ?, ?]. There exists a smooth space $\widetilde{\mathcal{M}}$, whose irreducible components are the unique smooth simply connected covering spaces of the components of $\mathcal{M}$ [?] that are determined entirely by the representation theory of the relevant $N=(4,4)$ superconformal algebra $\mathcal{A}$.

First, since $\mathcal{A}\supset su(2)_l\oplus su(2)_r$, one finds that each holonomy Lie algebra of $\widetilde{\mathcal{M}}$ must contain $su(2)\oplus su(2)\oplus o(4+\delta)$ for some $\delta\in\mathbb{N}$. Using Berger's classification [?] one concludes [?, ?] that each irreducible component $\widetilde{\mathcal{M}}^\delta$ of $\widetilde{\mathcal{M}}$ is a Grassmannian of oriented positive definite four planes in an $\mathbb{R}^{4,4+\delta}$,

$$\widetilde{\mathcal{M}}^\delta \cong \mathcal{T}^{4,4+\delta} \cong O^+(4,4+\delta)/\left(SO(4)\times O(4+\delta)\right). \tag{1.1}$$

Here, for a vector space W with scalar product $\langle\cdot,\cdot\rangle$, $O^+(W)$ consists of those elements of $O(W)$ which do not interchange the two components of the space of oriented maximal positive definite subspaces in W, and $O(a,b)=O(\mathbb{R}^{a,b})$ etc. In general, $\mathcal{T}^{a,b}$ denotes the Grassmannian of oriented maximal positive definite subspaces in $\mathbb{R}^{a,b}$.

Second, our assumptions on SCFTs parametrized by $\widetilde{\mathcal{M}}$ imply that we can associate an "elliptic genus" to each theory in $\widetilde{\mathcal{M}}$. In fact, the elliptic genus is an invariant for each component $\widetilde{\mathcal{M}}^\delta$ and is given by a theta function of degree 2 with fixed characteristic and normalization. There exist only two such functions, namely the (vanishing) geometric elliptic genus of a complex two torus ($\delta=0$) and that of a $K3$ surface ($\delta=16$). By the results of [?], there exist theories of either elliptic genus.

Therefore, the covering space of the moduli space of those SCFTs with $c=6$ which provide representations of $\mathcal{A}$ is $(\widetilde{\mathcal{M}}^0)^{\oplus N_0}\oplus(\widetilde{\mathcal{M}}^{16})^{\oplus N_{16}}$ with some nonzero $N_0,\ N_{16}\in\mathbb{N}$. One can prove $N_0=1$ (see, e.g., [?, Ths. 7.1.1, 7.1.2]), but $N_{16}=1$ remains a widely used conjecture. However, as we shall explain below, all theories that arise in string theory and are expected to exhibit connections to geometry are parametrized either by $\widetilde{\mathcal{M}}^0$ or by a single component of $(\widetilde{\mathcal{M}}^{16})^{\oplus N_{16}}$. The corresponding two-component moduli space $\mathcal{M}$, for simplicity, is dubbed MODULI SPACE OF $N=(4,4)$ SCFTs WITH CENTRAL CHARGE $c=6$ nevertheless:

Proposition 1.1. [?, ?, ?] *The moduli space $\mathcal{M}$ decomposes into two components $\mathcal{M}^{tori}=\mathcal{M}^0$, $\mathcal{M}^{K3}=\mathcal{M}^{16}$ with smooth simply connected covering spaces $\widetilde{\mathcal{M}}^\delta$, $\delta\in\{0,16\}$ as in* (1.1). *The assignment to either component is obtained by the elliptic genus which associates each theory to the torus or to $K3$.*

As mentioned in the Introduction, the theories in $\mathcal{M}$ are important in the context of string theory. In fact [?], in the case at hand we expect each theory parametrized by $\widetilde{\mathcal{M}}=\widetilde{\mathcal{M}}^0\oplus\widetilde{\mathcal{M}}^{16}$ to have a nonlinear sigma model realization which describes propagation of strings on some Calabi-Yau manifold X of complex dimension 2, i.e. on a complex two torus or a K3 surface. This justifies our (possible) restriction to the two-component $\mathcal{M}$ as in Prop. 1.1. The parameters of a nonlinear sigma model are an Einstein metric g on X (possibly in the orbifold limit), and a so-called B-field $B\in H^2(X,\mathbb{R})$. The metric g is uniquely determined by the volume $V\in\mathbb{R}^+$ of

X together with the three plane $\Sigma \subset H^2(X,\mathbb{R})$ that is invariant under the Hodge star operator for g, which acts as an involution on $H^2(X,\mathbb{R})$. On cohomology, we use the metric which is induced by the intersection pairing under Poincaré duality. Then we have $H^2(X,\mathbb{R}) \cong \mathbb{R}^{3,3+\delta}$ with $\delta \in \{0,16\}$ as in Prop. 1.1, and Σ is positive definite. In other words, the parameter space of nonlinear sigma models on X is

$$\begin{aligned} &\mathcal{T}^{3,3+\delta} \times \mathbb{R}^+ \times H^2(X,\mathbb{R}) \\ &\quad \cong O^+(3,3+\delta)/\left(SO(3)\times O(3+\delta)\right) \times \mathbb{R}^+ \times \mathbb{R}^{3,3+\delta}. \end{aligned} \tag{1.2}$$

Since (1.2) parametrizes theories with elliptic genus given by the geometric elliptic genus of X [**?**], and by Prop. 1.1, the spaces (1.2) must be isomorphic to the $\widetilde{\mathcal{M}}^\delta \cong \mathcal{T}^{4,4+\delta}$. Indeed, for $\delta = 16$ this was shown in [**?**], and the same technique works for $\delta = 0$. The explicit isomorphism depends on the choice of a null vector $\upsilon \in \mathbb{R}^{4,4+\delta}$. For a four plane $x \in \widetilde{\mathcal{M}}^\delta$ determine the three plane $\widehat{\Sigma} := x \cap \upsilon^\perp$ and $\xi_4 \in x \cap \widehat{\Sigma}^\perp$ with $\langle \xi_4, \upsilon\rangle = 1$. Note that $\upsilon^\perp/\upsilon \cong \mathbb{R}^{3,3+\delta}$; specify a projection $pr : \upsilon^\perp \to \mathbb{R}^{3,3+\delta} \subset \mathbb{R}^{4,4+\delta}$ by choosing another null vector $\upsilon^0 \in \mathbb{R}^{4,4+\delta}$ with $\langle \upsilon, \upsilon^0\rangle = 1$ and $pr(\upsilon^\perp) \perp \upsilon^0$. Then $\Sigma := pr(\widehat{\Sigma}),\ V := \xi_4^2/2,\ B := pr(\xi_4 - \upsilon^0)$ are the corresponding sigma model data.

This isomorphism depends on the projection from $\mathbb{R}^{4,4+\delta} \cong H^{even}(X,\mathbb{R})$ (without grading) onto $\mathbb{R}^{3,3+\delta} \cong H^2(X,\mathbb{R})$ (which fixes the grading). Hence $x \in \widetilde{\mathcal{M}}^\delta$ is naturally interpreted as four plane in $H^{even}(X,\mathbb{R})$, and υ, υ^0 as generators of $H^4(X,\mathbb{Z})$, $H^0(X,\mathbb{Z})$. By Poincaré duality the lattices $H^{even}(X,\mathbb{Z})$, $H^2(X,\mathbb{Z})$ are even unimodular of signature $(4,4+\delta)$, $(3,3+\delta)$, and therefore are uniquely determined up to automorphisms [**?**]. We assume that an embedding $H^{even}(X,\mathbb{Z}) \hookrightarrow H^{even}(X,\mathbb{R})$ has been fixed. Then

Theorem 1.2. *The smooth simply connected component $\widetilde{\mathcal{M}}^\delta,\ \delta \in \{0,16\}$, of the cover $\widetilde{\mathcal{M}}$ of the moduli space $\mathcal{M}$, which is associated to a complex two torus or a K3 surface X, respectively, is given by the Grassmannian of oriented positive definite four planes in $H^{even}(X,\mathbb{R})$. The position of $x \in \widetilde{\mathcal{M}}^\delta$ is specified by its relative position with respect to the lattice $H^{even}(X,\mathbb{Z})$. $\widetilde{\mathcal{M}}^\delta$ is isomorphic to the parameter space* (1.2) *of nonlinear sigma models on X. The explicit isomorphism depends on the choice of two null vectors $\upsilon,\ \upsilon^0 \in H^{even}(X,\mathbb{Z})$ with $\langle \upsilon, \upsilon^0\rangle = 1$:*

$$\begin{aligned} \mathcal{T}^{4,4+\delta} \cong \widetilde{\mathcal{M}}^\delta \ni x \quad &\longmapsto \quad (\Sigma, V, B) \ \in\ \mathcal{T}^{3,3+\delta} \times \mathbb{R}^+ \times H^2(X,\mathbb{R}), \\ x \quad &= \mathrm{span}_{\mathbb{R}}\left\{ \xi(\Sigma)\,,\ \xi_4(V,B) = \upsilon^0 + B + (V - B^2 2)\upsilon \right\}, \\ &\quad \textit{where for } \sigma \in \Sigma,\ \ \xi(\sigma) := \sigma - \langle B,\sigma\rangle\upsilon. \end{aligned} \tag{1.3}$$

The three plane $\Sigma \subset H^2(X,\mathbb{R})$, which together with the volume $V \in \mathbb{R}^+$ determines an Einstein metric on X, is specified by its relative position with respect to $H^2(X,\mathbb{Z})$.

The components $\mathcal{M}^\delta,\ \delta \in \{0,16\}$, of the moduli space $\mathcal{M}$ are now obtained from $\widetilde{\mathcal{M}}^\delta,\ \delta \in \{0,16\}$, by modding out appropriate discrete groups. These are the groups of equivalences of SCFTs with different nonlinear sigma model parameters, or in physicists' terminology the T-DUALITY GROUPS. In fact, by [**?**, **?**, **?**] the appropriate groups are just the orientation preserving lattice automorphisms of our reference lattices $H^{even}(X,\mathbb{Z})$. They act transitively on pairs (υ,υ^0) of null vectors in $H^{even}(X,\mathbb{Z})$ with $\langle \upsilon, \upsilon^0\rangle = 1$:

Theorem 1.3. *The component $\mathcal{M}^\delta$, $\delta \in \{0, 16\}$, of the moduli space $\mathcal{M}$, which is associated to a complex two torus or a $K3$ surface X, is given by*

$$\mathcal{M}^\delta = O^+(H^{even}(X,\mathbb{Z}))\backslash O^+(H^{even}(X,\mathbb{R}))/SO(4) \times O(4+\delta).$$

Summarizing, we have an explicit description of our moduli space $\mathcal{M}$ of $N = (4,4)$ SCFTs with $c = 6$: The two smooth simply connected components of its cover $\widetilde{\mathcal{M}}$ can be understood as extensions of the "geometric" Teichmüller spaces of Einstein metrics (including orbifold limits) on a torus or $K3$ surface X by the additional parameters of B-fields $B \in H^2(X,\mathbb{R})$. In particular, $\mathcal{M}$ is the moduli space of such SCFTs with central charge $c = 6$ which are representations of $\mathcal{A}$ and admit nonlinear sigma model descriptions. Let us remark that to date, no SCFT with $c = 6$ and superconformal algebra $\mathcal{A}$ has been found not to belong to $\mathcal{M}$. The parameters of a SCFT in $\widetilde{\mathcal{M}}$ are encoded in a positive definite four plane $x \subset H^{even}(X,\mathbb{R})$ which is specified by its relative position with respect to the lattice $H^{even}(X,\mathbb{Z}) \subset H^{even}(X,\mathbb{R})$. Each choice of null vectors υ, $\upsilon^0 \in H^{even}(X,\mathbb{Z})$ with $\langle \upsilon, \upsilon^0 \rangle = 1$ is interpreted as choice of generators of $H^4(X,\mathbb{Z})$, $H^0(X,\mathbb{Z})$, respectively. It fixes the projection $H^{even}(X,\mathbb{R}) \to H^2(X,\mathbb{R})$ and thereby a GEOMETRIC INTERPRETATION (Σ, V, B) of x. If $B = 0$ in such a geometric interpretation, then x is the $+1$ eigenspace in $H^{even}(X,\mathbb{R})$ of the Hodge star operator which corresponds to the Einstein metric given by (Σ, V). All equivalences of SCFTs in $\widetilde{\mathcal{M}}$ are lattice automorphisms in $O^+(H^{even}(X,\mathbb{Z}))$.

In the Introduction, we have mentioned that $\mathcal{M}$ is a partial completion of the actual moduli space of SCFTs we are interested in. Namely, $\mathcal{M}^{K3}$ contains points which do not correspond to well-defined SCFTs [**?**]. They form subvarieties of $\mathcal{M}^{K3}$ with at least complex codimension 1 [**?**]. These ill-behaved theories, however, will not be of relevance for the discussion below.

2. Orbifold conformal field theories on $K3$

In the previous section, we have described the moduli space $\mathcal{M}$ of $N = (4,4)$ SCFTs with $c = 6$ as algebraic space. Every theory in its toroidal component $\mathcal{M}^{tori}$ can be explicitly constructed as a toroidal SCFT [**?**]. Anything but an analogous statement is true for $\mathcal{M}^{K3}$, however, due to the fact that no smooth Einstein metric on $K3$ is known explicitly. We can only construct a finite subset of theories in $\mathcal{M}^{K3}$, which is given by Gepner models and orbifolds thereof (GEPNER TYPE MODELS), and CFTs in lower dimensional subvarieties obtained by an orbifold procedure from theories in appropriate subvarieties of $\mathcal{M}^{tori}$. The description of these subvarieties of $\mathcal{M}^{K3}$ is the object of the present section.

Given a SCFT $\mathcal{C}$ and a finite group G that acts on its Hilbert space, under certain additional assumptions on this action one can construct a well-defined ORBIFOLD CONFORMAL FIELD THEORY $\mathcal{C}/G$. It is obtained by projecting onto G invariant representations of the superconformal algebra and adding so-called TWISTED REPRESENTATIONS[1], each of which is nonlocal with respect to a representation in the original theory $\mathcal{C}$.

In the case of interest to us, where $\mathcal{C} = \mathcal{C}_T$ is a fixed toroidal SCFT corresponding to some $x_T \in \mathcal{M}^{tori}$, all assumptions on the G action are fulfilled if we can find a geometric interpretation (Σ_T, V_T, B_T) for x_T on a G symmetric torus T with metric given by (Σ_T, V_T) such that $B_T \in H^2(T,\mathbb{R})^G$ and the action of G is induced by the

[1] See Sect. 4 for further details.

geometric action on T. These data are assumed to be fixed in the following. We suppose that G acts non-trivially on T and does not contain non-trivial translations. Moreover, we assume that all singularities $s \in \mathcal{S} \subset T/G$ can be minimally resolved to obtain a $K3$ surface $p : X = \widetilde{T/G} \to T/G$, which in particular implies $G \subset SU(2)$. Such G actions have been classified [?], and the relevant groups are cyclic, binary dihedral or tetrahedral, respectively:

$$\mathbb{Z}_N,\ N \in \{2,3,4,6\}, \quad \widehat{D}_n,\ n \in \{4,5\}, \quad \widehat{\mathbb{T}}. \tag{2.1}$$

As to notations, let $\pi : T \to X$ denote the induced rational map of degree $|G|$ which is well defined away from the fixed points of G. Each singularity $s \in \mathcal{S} \subset T/G$ is of ADE type, such that the intersection matrix for the irreducible components of the exceptional divisor $p^{-1}(s)$ is given by the negative of the Cartan matrix of the ADE group corresponding to the singularity s. The Poincaré duals of the $(n_s - 1)$ components of $p^{-1}(s)$ are denoted $\widehat{E}_s^{(l)},\ l \in \{1, \ldots, n_s - 1\}$. Moreover, the $\mathbb{Z}$-span of these cocycles is denoted $\widehat{\mathbb{E}}_s \subset H^2(X, \mathbb{Z})$, and $\widehat{\mathbb{E}} := \oplus_s \widehat{\mathbb{E}}_s$. In writing $X = \widetilde{T/G}$ we mean the orbifold limit of $K3$, i.e. X is equipped with the metric induced by the flat metric (Σ_T, V_T) on T which assigns volume zero to each cycle corresponding to an $\widehat{E}_s^{(l)} \in \widehat{\mathbb{E}}$.

By calculating the elliptic genus it is not hard to check that the orbifold CFT $\mathcal{C}_T/G$ belongs to the $K3$ component $\mathcal{M}^{K3}$ of the moduli space. In the following, we will in fact assume that it has a geometric interpretation on $X = \widetilde{T/G}$. How such a geometric interpretation (Σ, V, B) is consistently obtained shall be explained in two steps. First, the metric on X has to be specified by giving the volume V of X and $\Sigma \subset H^2(X, \mathbb{R})$. Second, we need to determine the B-field $B \in H^2(X, \mathbb{R})$ of $\mathcal{C}_T/G$.

For the first step, the "geometric" part of the problem, we have $V_T = |G|V$, and we know that Σ will be specified by its relative position with respect to $H^2(X, \mathbb{Z})$. Since the relative position of Σ_T with respect to $H^2(T, \mathbb{Z})^G$ is known, the strategy is to determine the embedding $\pi_* H^2(T, \mathbb{Z})^G \hookrightarrow H^2(X, \mathbb{Z})$. This can be done by generalizing methods due to Nikulin, who in [?] considered the case $G = \mathbb{Z}_2$. To this end, note that $\pi_* H^2(T, \mathbb{Z})^G \perp \widehat{\mathbb{E}}$ is a sublattice of maximal rank in $H^2(X, \mathbb{Z})$. Moreover[2], by [?] one has $\pi_* H^2(T, \mathbb{Z})^G \cong H^2(T, \mathbb{Z})^G(|G|)$ which we implement by identifying $\pi_* H^2(T, \mathbb{Z})^G \cong \{\sqrt{|G|}\kappa \mid \kappa \in H^2(T, \mathbb{Z})^G\}$, where for $\kappa \in H^2(T, \mathbb{Z})^G$ we keep on using the original scalar product. The key observation is that it suffices to find the maximal primitive sublattices $K_{|G|} \subset \pi_* H^2(T, \mathbb{Z})^G$, $\widehat{\Pi}_{|G|} \subset \widehat{\mathbb{E}}$ in $H^2(X, \mathbb{Z})$ and apply the following Th. 2.1. It allows to describe the lattice $H^2(X, \mathbb{Z})$ in terms of the sublattices $K_{|G|}$, $\widehat{\Pi}_{|G|}$:

Theorem 2.1. [?, Prop.1.6.1], [?, §1] *Let Λ denote a primitive nondegenerate sublattice of an even unimodular lattice Γ and Λ^* its dual, with $\Lambda \hookrightarrow \Lambda^*$ by use of the metric on Λ. Then the embedding $\Lambda \hookrightarrow \Gamma$ with $\Lambda^\perp \cap \Gamma \cong \mathcal{V}$ is specified by an isomorphism $\gamma : \Lambda^*/\Lambda \to \mathcal{V}^*/\mathcal{V}$, such that the induced quadratic forms $q_\Lambda : \Lambda^*/\Lambda \to \mathbb{Q}/2\mathbb{Z}$, $q_\mathcal{V} : \mathcal{V}^*/\mathcal{V} \to \mathbb{Q}/2\mathbb{Z}$ obey $q_\Lambda = -q_\mathcal{V} \circ \gamma$. Moreover,*

$$\Gamma \cong \left\{ (\lambda, \vee) \in \Lambda^* \oplus \mathcal{V}^* \mid \gamma(\overline{\lambda}) = \overline{\vee} \right\},$$

where $\overline{l}$ denotes the projection of $l \in L^*$ onto L^*/L.

[2]Given a lattice Γ, by $\Gamma(n)$ one denotes the lattice which agrees with Γ as a $\mathbb{Z}$ module but has quadratic form scaled by a factor of n.

In [**?**] Nikulin showed that for $G = \mathbb{Z}_2$ one has $K_2 \cong H^2(T,\mathbb{Z})(2)$, and $\widehat{\Pi}_2$ is the KUMMER LATTICE [**?**]. Let $\mu_1,\dots,\mu_4$ denote generators of $H^1(T,\mathbb{Z})$ and $Q^k_{i,j},\ i,j,k\in\{1,\dots,4\}$, the $\mathbb{Z}_2$ invariant representatives of the Poincaré dual of $2\mu_i\wedge\mu_j$ with $\mathcal{S}\cap(Q^k_{i,j}/\mathbb{Z}_2)\neq\varnothing$. Then one finds

$$\begin{aligned} H^2(X,\mathbb{Z}) \;&=\; \mathrm{span}_{\mathbb{Z}}\Big\{\sqrt{2}\mu_i\wedge\mu_j\in K_2;\quad \epsilon\in\widehat{\mathbb{E}};\\ &\qquad 1\sqrt{2}\mu_i\wedge\mu_j + 12\sum_{s\in\mathcal{S}\cap(Q^k_{i,j}/\mathbb{Z}_2)}\widehat{E}_s,\quad i,j,k\in\{1,\dots,4\}\Big\}. \end{aligned} \tag{2.2}$$

In [**?**], a description analogous to (2.2) was given for all cases under discussion. In particular, it suffices to determine the pairs $\kappa+\epsilon\in H^2(X,\mathbb{Z})$ with $\kappa\in K^*_{|G|}$ and $\epsilon\in\widehat{\Pi}^*_{|G|}$ but nonzero $\overline{\kappa},\ \overline{\epsilon}$. Their $\widehat{\Pi}^*_{|G|}$ contributions $\overline{\epsilon}$ are always given by linear combinations of $1|G|\widehat{E}^{(l)}_s$ with $s\in\mathcal{S}\cap(Q/G)$ for an appropriate representative Q of the Poincaré dual of $\pi^*(\kappa)$. From [**?**, Prop.2.1] one obtains[3] all $K_{|G|},\ \widehat{\Pi}_{|G|}$ as well as the embeddings $\pi_*H^2(T,\mathbb{Z})^G\hookrightarrow H^2(X,\mathbb{Z})$. Hence, we can specify the relative position of $\Sigma:=\pi_*\Sigma_T$ with respect to $H^2(X,\mathbb{Z})$, and (Σ,V) specifies the Einstein metric on X.

The second step in the determination of (Σ,V,B) uses the same idea as the first one. In [**?**] we show that the images $\sqrt{|G|}\upsilon,\ \sqrt{|G|}\upsilon^0$ of $\upsilon,\ \upsilon^0$ in $H^{even}(X,\mathbb{Z})$ generate a primitive sublattice. Moreover, the maximal primitive sublattice $\widehat{K}_{|G|}$ of $\pi_*H^{even}(T,\mathbb{Z})^G$ in $H^{even}(X,\mathbb{Z})$ obeys

$$\widehat{K}^*_{|G|}/\widehat{K}_{|G|}\cong K^*_{|G|}/K_{|G|}\times\mathbb{Z}^2_{|G|}\cong\widehat{\Pi}^*_{|G|}/\widehat{\Pi}_{|G|}\times\mathbb{Z}^2_{|G|}.$$

Hence Th. 2.1 implies that $\widehat{K}_{|G|},\ \widehat{\Pi}_{|G|}$ cannot be embedded in $H^{even}(X,\mathbb{Z})$ as orthogonal sublattices. Rather, instead of $\sqrt{|G|}\upsilon$ and $\sqrt{|G|}\upsilon^0$, appropriate generators of $H^4(X,\mathbb{Z})$ and $H^0(X,\mathbb{Z})$ are

$$\widehat{\upsilon}:=\sqrt{|G|}\,\upsilon,\qquad \widehat{\upsilon}^0:=1\sqrt{|G|}\upsilon^0-1|G|\widehat{B}_{|G|}-\widehat{\upsilon},\quad \widehat{B}_{|G|}\in\widehat{\Pi}_{|G|}, \tag{2.3}$$

and $\widehat{B}_{|G|}$ is uniquely determined up to irrelevant lattice automorphisms [**?**, Lem. 3.2]. This gives $\pi_*H^{even}(T,\mathbb{Z})^G\hookrightarrow H^{even}(X,\mathbb{Z})$; since the relative position of x_T with respect to $H^{even}(T,\mathbb{Z})^G$ is known, it allows to read off the desired geometric interpretation of the orbifold CFT corresponding to $x=\pi_*x_T\in\mathcal{M}^{K3}$. In particular, B is obtained[4] by rewriting the vector $\xi_4(V_T,B_T)$ of (1.3) in terms of $\widehat{\upsilon},\ \widehat{\upsilon}^0$ instead of $\upsilon,\ \upsilon^0$:

Proposition 2.2. [**?**, Th. 3.3] *Let (Σ_T,V_T,B_T) denote a geometric interpretation of a toroidal SCFT $x_T\in\mathcal{M}^{tori}$ on the torus T that admits a G symmetry, $G\subset SU(2)$ not containing non-trivial translations; all possible G are listed in* (2.1). *Then its image $x\in\mathcal{M}^{K3}$ under the G orbifold procedure has geometric interpretation (Σ,V,B) where $\Sigma=\pi_*\Sigma_T$, $V=V_T|G|$, and $B=1\sqrt{|G|}B_T+1|G|\widehat{B}_{|G|}$. Here, $\widehat{B}_{|G|}\in\widehat{\Pi}_{|G|}$, and for each $G_s\subset G$ type singularity $s\in\mathcal{S}$ and $l\in\{1,\dots,n_s-1\}$, $\langle\widehat{B}_{|G|},\widehat{E}^{(l)}_s\rangle$ is the $|G\colon G_s|$–fold coefficient of $\widehat{E}^{(l)}_s$ in the highest root of $\widehat{\mathbb{E}}_s$.*

[3] For cyclic groups $G=\mathbb{Z}_N,\ N\in\{3,4,6\}$, $\widehat{\Pi}_N$ was first found in [**?**].

[4] For $G=\mathbb{Z}_2$, the correct B-field was first found in [**?**]; using D-geometry, it was determined for all cyclic groups in [**?**, **?**]; together with W. Nahm in [**?**] we rederive it for $\mathbb{Z}_2$ and $\mathbb{Z}_4$.

Summarizing, for the G orbifold of a toroidal theory $\mathcal{C}_T$ in $\mathcal{M}^{tori}$ the choice of geometric interpretation on the orbifold limit $\widetilde{T/G}$ of $K3$ induces a certain nonzero fixed B-field $1|G|\widehat{B}_{|G|}$ in direction of the exceptional divisor of the blow up, as predicted in [?]. This B-field can be determined explicitly by classical geometric considerations. In particular, (2.3) implies

$$|G|v^0 = \pi^* \left(|G|\,\widehat{v}^0 + \widehat{B}_{|G|} + |G|\widehat{v} \right), \tag{2.4}$$

which allows for an interpretation [?] in terms of the classical McKay correspondence [?, ?]. Namely, we interpret (2.4) as an equation of Mukai vectors for vector bundles on T, X. More precisely, $|G|v^0$ corresponds to a trivial bundle of rank $|G|$ on T that naturally carries the regular representation of G on the fibers, yielding a G equivariant flat bundle. By the results in [?], there is a corresponding bundle on X whose Mukai vector should be given by the argument of π^* in (2.4). In particular, $\widehat{B}_{|G|}$ must be the first Chern class of that bundle, which receives a contribution $|G\colon G_s|\widehat{B}^s_{|G|}$ from each $G_s \subset G$ type singularity $s \in \mathcal{S}$ that is determined by the classical McKay correspondence [?, ?, ?]: According to the decomposition $\rho_s = \sum_l m_s^{(l)} \rho^{(l)}$ of the regular representation ρ_s of G_s into irreducible ones, we have $\widehat{B}^s_{|G|} = \sum_l m_s^{(l)} (\widehat{E}_s^{(l)})^*$, where $\sum_l m_s^{(l)} \widehat{E}_s^{(l)}$ is the highest root of $\widehat{\mathbb{E}}_s$, and $\{(\widehat{E}_s^{(l)})^*\} \subset \mathbb{E}_s \otimes \mathbb{Q}$ denotes the dual basis of the fundamental system $\{\widehat{E}_s^{(l)}\}$ of $\widehat{\mathbb{E}}_s$. This is in exact agreement with Prop. 2.2. For cyclic G we can calculate the entire Mukai vector that is expected on the right hand side of (2.4) and find agreement.

3. Recreational interlude on mirror symmetry

Since in Sect. 5 we will compare various approaches to mirror symmetry on $K3$ in a particular example, the present section is devoted to a sketchy overview of basic ideas of mirror symmetry. We apologize for the inevitable incompleteness but refer the reader to the literature for details (see e.g. [?]).

We view mirror symmetry as an equivalence of $N = (2,2)$ SCFTs induced by the outer automorphism of the left handed $N = 2$ superconformal algebra which inverts the sign of the $U(1)$ current and interchanges the two supercharges [?, ?]. If the equivalent SCFTs admit (at least approximate) geometric interpretations as nonlinear sigma models on different Calabi-Yau manifolds X, X', mirror symmetry must induce a "nonclassical duality" between geometrically unrelated manifolds, an observation that has had striking impact on both mathematics and physics. Meanwhile, it has become a well-known slogan that mirror symmetry interchanges complex and (quantum corrected) complexified Kähler moduli of X, X'.

The first explicit construction of mirror dual SCFTs was given by Greene and Plesser in [?]. For each Gepner type model $\mathcal{C}/H$ obtained as Abelian H orbifold from a Gepner model $\mathcal{C}$ with central charge $c = 3d,\ d \in \mathbb{N}$, they determine an Abelian group H^* of symmetries[5] of $\mathcal{C}$ such that the orbifold $\mathcal{C}/H^*$ is the mirror of $\mathcal{C}/H$. A specific subgroup G of the symmetry group of a Gepner type model can be used to characterize the family of Calabi-Yau manifolds which admit an algebraic G action [?, ?]. These Calabi-Yau manifolds should be lowest order approximations to string vacua constructed from the corresponding Gepner type model. This enabled

[5] We call H^* the Greene/Plesser (GP) group. For further details, see the proof of Prop. 5.1.

Greene and Plesser to give a geometric meaning to their version of mirror symmetry as a duality between families of Calabi-Yau manifolds, with highly non-trivial verification found in [**?**, **?**].

The idea to characterize families of Calabi-Yau manifolds by their algebraic automorphisms translates nicely into the construction of families of Calabi-Yau hypersurfaces in toric varieties. Indeed, the "toric description" of mirror symmetry states that the family of Calabi-Yau toric hypersurfaces corresponding to the reflexive polyhedron Δ has a mirror dual corresponding to the dual polyhedron Δ^* [**?**].

In Sect. 1 we have argued that each theory in our moduli space $\mathcal{M}$ admits nonlinear sigma model realizations on a complex two torus or a $K3$ surface X. Moreover, from Th. 1.3 we know that all equivalences of theories parametrized by $\widetilde{\mathcal{M}}$ are given by lattice automorphisms in $O^+(H^{even}(X,\mathbb{Z}))$. It is therefore natural to ask which of these automorphisms should describe mirror symmetry. The question is delicate since all theories in $\widetilde{\mathcal{M}}$ have enhanced supersymmetry beyond $N=(2,2)$, so to address the outer automorphism on the superconformal algebra $\mathcal{A}$ which inverts the sign of the left handed $U(1)$ current requires the choice of a Cartan torus in $su(2)_l \subset \mathcal{A}$. In terms of geometric interpretations this corresponds to the choice of a complex structure within the $\mathbb{S}^2$ of complex structures compatible with a given Einstein metric.

A solution to this problem that is closely related to the results on mirror symmetry in the context of toric geometry has been proposed by Aspinwall and Morrison [**?**]. To this end, let us concentrate on $\mathcal{M}^{K3}$. We consider a family $\{x_t\} \subset \widetilde{\mathcal{M}}^{K3}$ of so-called M POLARIZED theories, where $M \subset H^2(X,\mathbb{Z})$ is a primitive sublattice of signature $(1,\rho-1)$, $\rho \geq 1$. Namely, it is assumed that for each x_t a geometric interpretation (Σ_t, V_t, B_t), and also a compatible complex structure have been chosen such that M can be embedded as primitive sublattice into the corresponding Picard lattice, and then $B_t \in M \otimes \mathbb{R}$. The family of $\check{M}$ polarized theories is a mirror dual family iff there is an embedding of $M \perp \check{M}$ as sublattice of maximal rank into the unique (up to lattice automorphisms) even unimodular lattice $\Gamma^{2,18} \subset H^2(X,\mathbb{Z})$ of signature $(2,18)$. This construction has been discussed in detail by Dolgachev [**?**], where he also explains that Arnol'd's strange duality [**?**, **?**, **?**] actually is the oldest version of mirror symmetry ever investigated. The existence and uniqueness of $\check{M}$, however, cannot be proven in general, since an embedding $M \hookrightarrow \Gamma^{2,18} \subset H^2(X,\mathbb{Z})$ might not exist (uniquely).

The characterization of a family of Calabi-Yau manifolds by its generic Picard lattice is related to a characterization by its generic algebraic automorphisms. It is therefore natural to expect that this approach to mirror symmetry should have a toric cousin. Indeed, for any family of hypersurfaces in a toric variety corresponding to a reflexive polyhedron Δ we can determine the generic Picard lattice $Pic(\Delta)$. Moreover, if in the $K3$ case all divisors in $Pic(\Delta)$ are toric, $Pic(\Delta^*) \cong \check{Pic}(\Delta)$. This, however, is not the case in general. Instead, let $Pic_{tor}(\Delta) \subset Pic(\Delta)$ denote the sublattice of toric divisors in $Pic(\Delta)$, then Rohsiepe found [**?**, **?**] $Pic(\Delta^*) \cong \check{Pic}_{tor}(\Delta)$ and $Pic_{tor}(\Delta^*) \cong \check{Pic}(\Delta)$. Therefore, one arrives at the following more refined statement about mirror symmetry on $K3$:

Proposition 3.1. [**?**, Prop.4.1] *Let* $\{x_t\} \subset \widetilde{\mathcal{M}}^{K3}$ *denote the family of theories with geometric interpretation* (Σ_t, V_t, B_t)*, such that* $(\Sigma_t,\ V_t)$ *specifies the family of hypersurfaces in a toric variety corresponding to the reflexive polyhedron* Δ*,* *and* $B_t \in Pic_{tor}(\Delta) \otimes \mathbb{R}$. *Then there is a mirror family with geometric interpretation* $(\check{\Sigma}_t, \check{V}_t, \check{B}_t)$ *with* $(\check{\Sigma}_t, \check{V}_t)$ *corresponding to the dual polyhedron* Δ^* *and* $\check{B}_t \in Pic_{tor}(\Delta^*) \otimes \mathbb{R} = \check{Pic}(\Delta) \otimes \mathbb{R}$.

All geometric constructions of mirror symmetry mentioned so far refer to appropriate families of Calabi-Yau manifolds. In contrast, on the level of SCFTs the Greene/Plesser construction is a point to point map on $\widetilde{\mathcal{M}}$. We will argue that in favorable situations it is also possible to give a geometric point to point map for mirror symmetry, as expected from Th. 1.3. In fact, Sect. 5 is devoted to the discussion of an example where all versions of mirror symmetry on $K3$ can be calculated and compared, and we will show that they are equivalent there.

The basic idea goes back to Vafa and Witten [?]; consider a toroidal SCFT $\mathcal{C}_T$ with sigma model realization on a d dimensional complex torus T^{2d}. Projection onto the real part of each complex coordinate gives a torus fibration $T^{2d} \to T^d$ over a d dimensional real torus. In [?] it is shown that fiberwise T-duality in this fibration induces the mirror automorphism on the superconformal algebra of $\mathcal{C}_T$. Moreover, for a geometric symmetry G of T^{2d} which respects the fibration and supersymmetry one also obtains a mirror map for the orbifold $\mathcal{C}/G$. This idea has been extended in [?] to the celebrated Strominger/Yau/Zaslow (SYZ) conjecture which is supposed to hold for more general fibrations. It should be noted that for a generic member of an M polarized family of models in $\widetilde{\mathcal{M}}^{K3}$ the assumptions necessary for the SYZ mirror construction are fulfilled iff one can construct the mirror family à la Dolgachev [?, Cor.1.4].

In [?] together with W. Nahm we have used Vafa and Witten's approach to give an explicit construction of the mirror automorphism $\gamma_{MS} \in O^+(H^{even}(X,\mathbb{Z}))$ for $\mathbb{Z}_N$ orbifold CFTs in $\widetilde{\mathcal{M}}^{K3}$. First, using [?] for $\widetilde{\mathcal{M}}^{tori}$ we determine the appropriate lattice automorphism in $O^+(H^{even}(T,\mathbb{Z}))$ that is induced by fiberwise T-duality of $T \to T^2$. By making use of the $\mathbb{Z}_N$ orbifold constructions discussed in Sect. 2 we extend this automorphism to an element of $O^+(H^{even}(X,\mathbb{Z}))$ with $X = \widetilde{T/\mathbb{Z}_N}$, $N \in \{2,3,4,6\}$. In other words, with a suitable complex structure [?] we determine the mirror map which is induced by fiberwise T-duality in an elliptic fibration $X \to \mathbb{P}^1$ of the orbifold limit X of $K3$. In particular, we calculate the explicit action on the (non-stable) singular fibers of this fibration.

The extension of the lattice automorphism in $O^+(H^{even}(T,\mathbb{Z}))$ to an element of $O^+(H^{even}(X,\mathbb{Z}))$ is possible since by the discussion of Sect. 2 we know the explicit embeddings $\pi_* H^{even}(T,\mathbb{Z})^{\mathbb{Z}_N} \hookrightarrow H^{even}(X,\mathbb{Z})$. First recall that for $s \in \mathcal{S}$, $l \in \{1,\ldots,n_s-1\}$ the $\widehat{E}_s^{(l)} \in H^2(X,\mathbb{Z})$ are not orthogonal to $\widehat{K}_N \subset \pi_* H^{even}(T,\mathbb{Z})^{\mathbb{Z}_N}$. By Prop. 2.2

$$E_s^{(l)} := \widehat{E}_s^{(l)} - 1n_s\widehat{\upsilon} \tag{3.1}$$

are the orthogonal projections onto $\widehat{K}_N^\perp$ in $H^{even}(X,\mathbb{R})$. Analogously, Π_N denotes the orthogonal projection of $\widehat{\Pi}_N$ onto $\widehat{K}_N^\perp \cap H^{even}(X,\mathbb{R})$, and similarly for $\mathbb{E}_s$, $\mathbb{E}$. The mirror map must act as lattice automorphism on Π_N. In each case we have a description for $H^{even}(X,\mathbb{Z})$ analogous to (2.2). Knowing the mirror map on $\widehat{K}_N$ and applying it to lattice vectors of type $1\sqrt{N}\kappa + 1N\sum \widehat{E}_s^{(l)}$, $1\sqrt{N}\kappa \in K_N^*$, $\widehat{E}_s^{(l)} \in \widehat{\Pi}_N$, already determines the mirror map on Π_N up to automorphisms which are entirely under control [?]. For briefness of exposition, here we only state the formula for the action of mirror symmetry γ_{MS} on the Kummer lattice Π_2:

Proposition 3.2. *Consider a $\mathbb{Z}_N$ orbifold CFT on $K3$ constructed from a toroidal CFT with $\mathbb{Z}_N$ symmetric torus and vanishing B-field. For the $\mathbb{Z}_2$ orbifold limit of $K3$, we also assume the underlying torus to be orthogonal. Here, we use*[6] *$\mathcal{S} \cong \mathbb{F}_2^4$ to label the generators of $\mathbb{E}$ by $\mathbb{F}_2^4$, where the first two coordinates correspond to the fiber coordinates of the torus fibration. Then the version of mirror symmetry which is induced by fiberwise T-duality on the underlying toroidal theory acts on Π_2 by*

$$\forall\, (I,J,K,M) \in \mathbb{F}_2^4: \quad \gamma_{MS}(E_{(I,J,K,M)}) = 12 \sum_{i,j\in\mathbb{F}_2} (-1)^{iI+jJ} E_{(i,j,K,M)}.$$

*Similar formulas for the other $\mathbb{Z}_N$ orbifold limits of $K3$ are given in [**?**, (20)]. γ_{MS} is uniquely determined up to certain permutations of $\mathcal{S}$ and automorphisms of Π_N which are parametrized by $b \in \Pi_N/\mathbb{E}$ and act on the B-field by a shift by b.*

4. Conformal field theory versus Geometry

In the previous sections, we have mostly used geometric arguments to describe the moduli space $\mathcal{M}$ of $N=(4,4)$ SCFTs with $c=6$, its orbifold subvarieties and mirror symmetry for them. In the present section we will discuss corresponding results in CFT language in order to clarify the direct link between geometry and SCFT.

In Sect. 2 we have only briefly mentioned the construction of an orbifold SCFT $\mathcal{C}/G$ from a given theory $\mathcal{C}$ by projecting onto G invariant representations of the superconformal algebra and adding twisted ones. The lowest weight states of the latter are called TWISTED GROUND STATES and generate a finite dimensional subspace $\mathcal{T}$ of the Hilbert space of $\mathcal{C}/G$. A basis of $\mathcal{T}$ can be labeled $T_s^{[g]}$, where $[g],\ g \in G$, denotes a non-trivial conjugacy class in G, and $s \in \mathcal{S}_{[g]}$ accounts for degeneracies. If $\mathcal{C}$ has a nonlinear sigma model realization on some Calabi-Yau manifold Y and the G action is induced by a geometric action on it, then $\mathcal{S}_{[g]}$ consists of the G orbits in Y that are pointwise fixed by g.

Hence for the orbifold CFTs on $K3$ orbifold limits X of Sect. 2 the twisted ground states can be labeled $T_s^l,\ s \in \mathcal{S},\ l \in \{1,\ldots,n_s-1\}$. Their $1:1$ correspondence to components $\widehat{E}_s^{(l)}$ of exceptional divisors in the blow-up will be worked out in detail below. In fact, there is a standard scalar product $\langle\cdot,\cdot\rangle$ on the Hilbert space of each CFT with respect to which all states in the twisted representations are orthogonal to each state in the G invariant part of the original theory's Hilbert space. It is therefore even more natural to expect a $1:1$ correspondence between twisted ground states and the projections $E_s^{(l)}$ onto $\widehat{K}_{|G|}^{\perp}$, which for the $\mathbb{Z}_N$ orbifold CFTs will indeed be established below. We normalize the twisted ground states such that with ζ_n denoting a fixed primitive n^{th} root of unity

$$\begin{gathered}\forall s,\ s' \in \mathcal{S},\ l^{(\prime)} \in \{1,\ldots,n_{s^{(\prime)}}-1\}: \\ \left\langle T_s^l, T_{s'}^{l'} \right\rangle = \delta_{s,s'}\delta_{l,l'}\left(2-\zeta_{n_s}^l-\zeta_{n_s}^{-l}\right).\end{gathered} \tag{4.1}$$

From now on, we will restrict to the case of $\mathbb{Z}_N$ orbifold CFTs on $K3$ discussed in Sects. 2, 3. The underlying toroidal theory $\mathcal{C}_T$ is assumed to have a geometric interpretation on a four torus $T=\mathbb{R}^4/\Lambda$ with vanishing B-field. Here, $\Lambda \subset \mathbb{R}^4$ is a nondegenerate lattice of rank 4, and its dual is denoted Λ^*. In the $\mathbb{Z}_2$ case we assume Λ to be orthogonal. The toroidal theory $\mathcal{C}_T$ then possesses so-called

[6] As usual, $\mathbb{F}_p$ with p prime denotes the unique finite field with p elements.

VERTEX OPERATORS which are parametrized by $\Lambda^*\oplus\Lambda$ and also act on the twisted ground states of $\mathcal{C}_T/\mathbb{Z}_N$. As to notations, we use $\mathcal{S}\hookrightarrow 1N\Lambda/\Lambda$ and denote by θ the generator of the geometric $\mathbb{Z}_N$ action on T which naturally acts on Λ as well. The action of the vertex operators then induces the following representation W of $\Lambda^*\oplus\Lambda$ on $\mathcal{T}$ by restriction to leading order terms in the OPE (see [?, Sect. 5] for details):

$$\forall\,(\mu,\lambda)\in\Lambda^*\oplus\Lambda;\quad \forall\, s\in\mathcal{S}, l\in\{1,\dots,n_s-1\}:$$

$$W(\mu,\lambda)T_s^l \;=\; \zeta_N^{l\mu(Ns)}T_{s'}^l,\quad s'=s+(\mathbb{1}-\theta)^{-1}\lambda=s-1n_s\sum_{k=1}^{n_s-1}k\theta^k\lambda. \tag{4.2}$$

In particular, we have

$$\begin{aligned}&\forall\, q,q'\in\Lambda^*\oplus\Lambda,\quad \forall\, s\in\mathcal{S}, l\in\{1,\dots,n_s-1\}:\\ &W(q')W(q)T_s^l \;=\; \zeta_{n_s}^{l\phi_{n_s}(q,q')}W(q)W(q')T_s^l,\\ &\qquad\qquad \phi_n((\mu,\lambda),(\mu',\lambda')):=\sum_{k=1}^{n}k\left(\mu\theta^k\lambda'-\mu'\theta^k\lambda\right).\end{aligned}$$

In other words, W is a Weyl algebra representation of $\Lambda^*\oplus\Lambda$ on $\mathcal{T}$, where Λ^* acts diagonally by introducing phases, and Λ acts by translation on the base point $s\in\mathcal{S}$.

The action of (fiberwise) T–duality on the vertex operators of the toroidal theory $\mathcal{C}_T$ is given by an exchange of rank 2 sublattices of Λ^* and Λ. By using the above Weyl algebra representation and re-diagonalizing appropriately it is therefore possible to derive the effect of (fiberwise) T–duality on twisted ground states in $\mathcal{T}$. This was performed in joint work with W. Nahm in [?]. Again, for briefness of exposition we only state the result for $G=\mathbb{Z}_2$ explicitly:

Proposition 4.1. *Consider a $\mathbb{Z}_N$ orbifold CFT on $K3$ constructed from a toroidal SCFT with $\mathbb{Z}_N$ symmetric torus and vanishing B-field. For the $\mathbb{Z}_2$ case we also assume the underlying torus to be orthogonal. Then the action of fiberwise T-duality on the underlying toroidal theory induces a $\mathbb{Z}_N$ type fiberwise Fourier transform F on the twisted ground states in $\mathcal{T}$. Explicitly, for $\mathbb{Z}_2$ we have*

$$\forall\,(I,J,K,M)\in\mathbb{F}_2^4:\quad F(T_{(I,J,K,M)})=12\sum_{i,j\in\mathbb{F}_2}(-1)^{iI+jJ}T_{(i,j,K,M)}.$$

Similar formulas for $N\in\{3,4,6\}$ are given in [?, (29)].

Comparison with the geometric mirror map in Prop. 3.2 now allows to explicitly relate twisted ground states to the generators $E_s^{(l)}$ of $\Pi_N\otimes\mathbb{Q}$:

Proposition 4.2. [?] *As before, let γ_{MS}, F denote the action of mirror symmetry as induced by fiberwise T-duality on the Kummer type lattice Π_N and the space of twisted ground states $\mathcal{T}$, respectively. The $\mathbb{C}$ linear map*

$$C:\Pi_N\otimes\mathbb{C}\to\mathcal{T},\quad \forall\, s\in\mathcal{S},\ l\in\{1,\dots,n_s-1\}:\quad C(E_s^{(l)})=1\sqrt{n_s}\sum_{k=1}^{n_s-1}\zeta_{n_s}^{lk}T_s^k$$

obeys $FC=C\gamma_{MS}$. Moreover, if on cohomology we use the scalar product induced by the intersection form and on the Hilbert space of the orbifold CFT we use the standard scalar product with (4.1), then C induces an anti-isometry. The image of Π_N under C is invariant under Hermitean conjugation.

To understand the map C it is useful to study the "quantum" $\mathbb{Z}_N$ symmetry of the orbifold CFT. It has generator ϑ which acts trivially on the untwisted sector and by $\vartheta T_s^l = \zeta_{n_s}^l T_s^l$ on twisted ground states. Performing the $\mathbb{Z}_N$ orbifold construction by this "quantum" symmetry on the orbifold CFT reproduces the original toroidal SCFT. Since we want to study geometric features of our orbifold CFT we write out the induced ϑ action on $\widehat{\Pi}_N$ instead of Π_N:

$$\forall\, s \in \mathcal{S}: \qquad \vartheta\left(\widehat{E}_s^{(l)}\right) = \begin{cases} \widehat{E}_s^{(l+1)} & \text{if } l < n_s - 1, \\ \widehat{\upsilon} - \sum\limits_{j=1}^{n_s-1} \widehat{E}_s^{(j)} & \text{if } l = n_s - 1. \end{cases}$$

In other words, the "quantum" $\mathbb{Z}_N$ symmetry of a $\mathbb{Z}_N$ orbifold CFT has its geometric counterpart in the $\mathbb{Z}_{n_s}$ symmetry of the extended Dynkin diagram $\widehat{A}_{n_s-1}$ for the irreducible components of the exceptional divisor over each $\mathbb{Z}_{n_s}$ type singularity.

This observation has a simple explanation[7] in terms of the group algebra of $\mathbb{Z}_N$, if we restrict to the local picture over each $s \in \mathcal{S}$. First, introduce an auxiliary state T_s^0 which is subject to (4.1) (and therefore is NOT the vacuum). Then $\mathbb{Z}_{n_s}$ symmetrically extend C by $C(\widehat{\upsilon}) := \sqrt{n_s}\, T_s^0$ to obtain

$$\begin{aligned} \forall\, l \in \{1, \ldots, n_s - 1\}: \qquad C(\widehat{E}_s^{(l)}) &= 1\sqrt{n_s} \sum_{k=0}^{n_s-1} \zeta_{n_s}^{lk} T_s^k, \\ C(\widehat{\upsilon} - \sum_{j=1}^{n_s-1} \widehat{E}_s^{(j)}) &= 1\sqrt{n_s} \sum_{k=0}^{n_s-1} T_s^k. \end{aligned}$$

The extended map has the form of a discrete Fourier transform on $\mathbb{Z}_{n_s}$, if it is interpreted as map on the group algebra $\mathbb{C}\mathbb{Z}_{n_s}$ of $\mathbb{Z}_{n_s}$ with T_s^l corresponding, say, to the conjugacy class of $\zeta_{n_s}^l$. Then up to normalization the n_s elements of $\mathbb{C}\mathbb{Z}_{n_s}$ listed above are just the idempotents in $\mathbb{C}\mathbb{Z}_{n_s}$, which are known to be related to the conjugacy classes by a discrete Fourier transform.

Next, let us investigate the action of $\Lambda^* \oplus \Lambda$ on cohomology which is induced by the Weyl algebra representation (4.2) under the map C. It proves useful to work with a dual basis $\varepsilon_s^{(l)}$ with respect to $\{\widehat{E}_s^{(l)},\ l \in \{1, \ldots, n_s - 1\};\ \widehat{\upsilon} - \sum_{j=1}^{n_s-1} \widehat{E}_s^{(j)}\}$: As before, denote by $\{(\widehat{E}_s^{(l)})^*\} \subset \widehat{\mathbb{E}}_s \otimes \mathbb{Q}$ the dual basis with respect to the fundamental system $\{\widehat{E}_s^{(l)}\}$ of $\widehat{\mathbb{E}}_s$, then

$$\varepsilon_s^{(l)} := \left\{ \begin{array}{ll} \widehat{\upsilon}^0 & \text{if } l = 0, \\ \widehat{\upsilon}^0 + (\widehat{E}_s^{(l)})^* & \text{if } 1 \le l < n_s, \end{array} \right\}, \quad \forall\, l \in \mathbb{Z}: \varepsilon_s^{(l+n_s)} = \varepsilon_s^{(l)}.$$

The induced action is given by

$$\forall\, (\mu, \lambda) \in \Lambda^* \oplus \Lambda, \quad \forall\, s \in \mathcal{S},\ l \in \{0, \ldots, n_s - 1\}:$$

$$W(\mu,\lambda)\varepsilon_s^{(l)} = \varepsilon_{s'}^{(\mu(n_s s)+l)}, \quad s' = s + (\mathbb{1} - \theta)^{-1}\lambda = s - 1n_s \sum_{k=1}^{n_s-1} k\theta^k \lambda. \tag{4.3}$$

Similarly to our explanation of Prop. 2.2, there is a straightforward interpretation [**?**] in terms of the classical McKay correspondence. Indeed, each of the vectors $\varepsilon_s^{(l)}$ is the $H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Q})$ part of the Mukai vector for one of the locally free sheaves near $s \in \mathcal{S}$ that were constructed in [**?**]. More precisely, identify $s \in \mathcal{S}$ with

[7] We thank David E. Berenstein for this comment.

the origin in $\mathbb{C}^2$ and consider $Y = \widetilde{\mathbb{C}^2/\mathbb{Z}_{n_s}}$. By [?, ?], for each $\mathbb{Z}_{n_s}$ equivariant flat line bundle on $\mathbb{C}^2$ there is a corresponding locally free sheaf on Y. We choose a fixed generator of $\mathbb{Z}_{n_s}$ and assume that on the fiber of the bundle over s it acts by $\chi_s^{(l)}(z) = \zeta_{n_s}^l z$. Then $(\widehat{E}_s^{(l)})^*$ is the first Chern class of the associated bundle on Y, and $\varepsilon_s^{(l)}$ is the part of its Mukai vector relevant to our discussion.

We can now interpret the Weyl algebra action (4.3) as an action on line bundles over X [?]. Indeed, it is the natural action of $\Lambda^* \oplus \Lambda$, where Λ acts by translations by elements of the torsion subgroup of the Jacobian torus of T on the underlying torus T, and Λ^* acts by tensorizing with fixed line bundles. Moreover, the indeterminacy in our formula for mirror symmetry in Prop. 3.2 translates directly into the freedom of choice of an origin in the affine space which parametrizes $\mathbb{Z}_N$ equivariant flat line bundles on the four torus [?].

Summarizing, in Prop. 4.2 we have established the explicit map between geometric and conformal field theoretic data which characterize orbifolds: In the local picture over each singularity $s \in \mathcal{S}$, we have a correspondence between twist fields $T_s^1, \ldots, T_s^{n_s-1}$ and the vectors $\varepsilon_s^{(1)}, \ldots, \varepsilon_s^{(n_s-1)}$ which induces a correspondence between non-trivial conjugacy classes in $\mathbb{Z}_{n_s}$ and a basis of $H^2(Y, \mathbb{Q})$, $Y = \widetilde{\mathbb{C}^2/\mathbb{Z}_{n_s}}$. Since we have worked with fixed choices for roots of unity this is just a realization of the "dual" McKay correspondence proven more generally by Ito and Reid [?]: For $R \in \mathbb{N}$ let μ_R denote the group of complex R^{th} roots of unity. Then for each finite subgroup G of $SL(n, \mathbb{C})$, there is a $1:1$ correspondence between junior conjugacy classes in $Hom(\mu_{|G|}, G)$ and a basis of $H^2(Y, \mathbb{Q})$, where Y is a minimal model of $\mathbb{C}^n/G$. Note that for our $\mathbb{Z}_{n_s} \subset SL(2, \mathbb{C})$ all non-trivial conjugacy classes in $Hom(\mu_{n_s}, \mathbb{Z}_{n_s})$ are junior.

We have been mostly working in the compact setting where $X = \widetilde{T/\mathbb{Z}_N}$ with $T = \mathbb{R}^4/\Lambda$. Here, we found that our CFT version of the dual McKay correspondence is compatible with the natural Weyl algebra representations of $\Lambda^* \oplus \Lambda$ on twisted ground states and $\mathbb{Z}_N$ equivariant flat line bundles on T, respectively.

5. An instructive example

In Sect. 3 we have given an overview of various approaches to mirror symmetry on $K3$: Greene/Plesser's construction (GP) for Gepner type models, Aspinwall/Morrison's or Dolgachev's approach (AMD) for M polarized families with Rohsiepe's improvement for Calabi-Yau hypersurfaces in toric varieties, and Vafa/Witten's SYZ like approach (VW) which we have applied to $\mathbb{Z}_N$ orbifold CFTs in $\mathcal{M}^{K3}$ [?], as explained in Sect. 4. Not only does each of these approaches refer to a fairly different setting, but also is a comparison almost impossible since e.g. the GP construction as such does not involve the choice of a specific geometric interpretation. In [?, ?, ?] Aspinwall and Morrison noted that a comparison does make sense for models that are invariant under mirror symmetry; first, they should be such under all applicable versions of mirror symmetry, and second one can directly compare the induced map on the fields of the relevant SCFTs in a chosen "reference geometric interpretation".

Aspinwall and Morrison use the Gepner model (1)(5)(40), which has trivial GP group, i.e. is GP mirror self dual. A translation into the AMD approach is possible [?, ?], which should prove GP=AMD [?] and thereby provide an element of $O^+(H^{even}(X, \mathbb{Z}))$ that is needed for the proof of Th. 1.3. Unfortunately, for this

Gepner model there exists no geometric interpretation in terms of a $\mathbb{Z}_N$ orbifold CFT of some toroidal theory. We will therefore work with a different model: Consider the Gepner model $(2)^4$; the GP group to be modded out to produce the mirror is $\mathbb{Z}_4^2$. Take its $\mathbb{Z}_2^2$ subgroup to define the Gepner type model $\widetilde{(2)}^4$ as $\mathbb{Z}_2^2$ orbifold of $(2)^4$. By [**?**, Th. 3.7] it possesses a geometric interpretation as $\mathbb{Z}_2$ orbifold CFT, see Prop. 5.3 below.

Proposition 5.1. *The Greene/Plesser group to be modded out from $(2)^4$ to construct the mirror for the Gepner type model $\widetilde{(2)}^4$ agrees with that to be modded out to construct $\widetilde{(2)}^4$. In other words, $\widetilde{(2)}^4$ is GP mirror self dual.*

Proof. Recall the Greene/Plesser construction [**?**]. For a Gepner model $\mathcal{C} = (k_1)\cdots(k_r)$ with central charge $3d,\ d \in \mathbb{N}$, let z_i denote the generator of the $\mathbb{Z}_{k_i+2}$ phase symmetry for its i^{th} minimal model factor. Recall that $\prod_i z_i$ acts trivially on $\mathcal{C}$, and that those $\prod_i z_i^{a_i}$ with $a_i \in \mathbb{N}$, $\prod_i \zeta_{k_i+2}^{a_i} = 1$ are called ALGEBRAIC. The mirror model of $\mathcal{C}/H$ with algebraic H then is $\mathcal{C}/H^*$, where H^* contains all algebraic $\prod_i z_i^{b_i}$ which for all $\prod_i z_i^{a_i} \in H$ obey $\prod_i \zeta_{k_i+2}^{a_i b_i} = 1$.

In our case with $\mathcal{C} = (2)^4$ one checks that both H and H^* are generated by $z_1^2 z_2^2$ and $z_1^2 z_3^2$, proving the assertion. □

It follows that $\widetilde{(2)}^4$ is just as good a model to study mirror symmetry on as (1)(5)(40). Let us check with the AMD approach:

Proposition 5.2. *Let $\Delta \subset \mathbb{R}^3$ denote the reflexive polyhedron which is associated to $\widetilde{(2)}^4$. Then Δ is related to its dual Δ^* by a $GL(3,\mathbb{Z})$ transformation. In other words, $\widetilde{(2)}^4$ is AMD mirror self dual.*

The corresponding families of toric Calabi-Yau hypersurfaces have generic (toric) Picard lattice[8]

$$(5.1)\qquad \begin{array}{rcl} Pic_{tor}(\Delta) \;=\; \check{Pic}(\Delta^*) &=& \langle 4\rangle \oplus \langle -2\rangle^6 \qquad\qquad\quad \cong\; Pic_{tor}(\Delta^*),\\ Pic(\Delta) \;=\; \check{Pic}_{tor}(\Delta^*) &=& \langle -4\rangle \oplus \langle -2\rangle^4 \oplus \mathcal{D}_6(-1) \oplus U \;\cong\; Pic(\Delta^*). \end{array}$$

Proof. The Abelian group action associated to the mirror of $(2)^4$ is the diagonal $\mathbb{C}^*$ action on $\mathbb{C}^4$. Additionally, for $\widetilde{(2)}^4$ there is a $\mathbb{Z}_2^2$ action which is specified by the generators $z_1^2 z_2^2$ and $z_1^2 z_3^2$ that we used in the proof of Prop. 5.1, where z_i now acts by multiplication with ζ_{k_i+2} on the i^{th} component of $\mathbb{C}^4$. This induces three independent relations on $\mathbb{C}^4$, and in $\mathbb{Z}^3 \subset \mathbb{R}^3$ yields Δ^* as convex hull of

$$\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}1\\0\\2\end{pmatrix}, \begin{pmatrix}-3\\-2\\-2\end{pmatrix} \quad\in\quad \mathbb{R}^3.$$

It is a regular tetrahedron with edges of length 2 in lattice units. One finds

$$(5.2)\qquad \begin{array}{rcl} \Delta &=& \text{conv. hull}\left\{ \begin{pmatrix}3\\-2\\-2\end{pmatrix}, \begin{pmatrix}-1\\2\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\2\end{pmatrix}, \begin{pmatrix}-1\\0\\0\end{pmatrix} \right\} = A\Delta^*,\\ A &=& \begin{pmatrix}3 & -2 & -2\\ -2 & 2 & 1\\ -2 & 1 & 2\end{pmatrix} \in GL(3,\mathbb{Z}), \end{array}$$

[8] For $n \in \mathbb{Z}$ the lattice of rank 1 generated by a vector of length squared n is denoted $\langle n\rangle := \mathbb{Z}(n)$. Moreover, $\mathcal{D}_{2k}$ denotes the lattice $\mathcal{D}_{2k} := \{x \in \mathbb{Z}^{2k} \mid \sum_i x_i \equiv 0(2)\}$, and U is the HYPERBOLIC LATTICE generated by two null vectors with scalar product one.

proving that $(\widetilde{2})^4$ is AMD mirror self dual.

(5.1) follows from Prop. 3.1 together with the methods of [?, ?, ?]. Namely, by Prop. 3.1 and the above,

$$Pic(\Delta^*) \cong Pic(\Delta) = \check{Pic}_{tor}(\Delta^*), \quad Pic_{tor}(\Delta^*) \cong Pic_{tor}(\Delta) = \check{Pic}(\Delta^*),$$

and $Pic_{tor}(\Delta)$ can be directly read off from the lattice vectors on edges of Δ. First, three appropriate lattice points have to be removed[9], since they correspond to mere coordinate transformations. Here, we can omit one vertex and two neighboring centers of edges. The remaining diagram provides a Dynkin type diagram for the generators of $Pic_{tor}(\Delta)$. With the techniques explained in [?, ?, ?] one checks that each vertex corresponds to a toric divisor of self-intersection number -2, whereas centers of edges correspond to toric divisors with self-intersection number -4. Each connecting line in our diagram corresponds to an intersection number 2. One then checks that the resulting lattice is $Pic_{tor}(\Delta)$ as given in (5.1).

By the results of [?, ?, ?] each lattice vector which is an inner point in an edge θ of Δ will correspond to a toric divisor on our $K3$ surface which splits into k disjoint divisors, where k is the length (in lattice units) of the edge θ^* of Δ^* which is dual to θ. If $k > 1$, all intersection numbers are divided by k to obtain the intersection numbers for these non-toric components. Therefore in our case each inner point of an edge corresponds to a toric divisor which splits into two disjoint non-toric ones with self-intersection number -2 each. With a somewhat lengthy calculation, the resulting lattice is checked to agree with $Pic(\Delta)$ as given in (5.1). □

To compare with the VW approach to mirror symmetry we need to have an appropriate geometric interpretation of $(\widetilde{2})^4$ as $\mathbb{Z}_N$ orbifold CFT of some toroidal theory. Indeed,

Proposition 5.3. *The Gepner type model $(\widetilde{2})^4$ admits a geometric interpretation as $\mathbb{Z}_2$ orbifold CFT of the toroidal model on $T = \mathbb{R}^4/\Lambda$, $\Lambda = 1\sqrt{2}\mathcal{D}_4$ with the B-field B^* given in* (5.3) *for which the theory has enhanced symmetry by the Frenkel-Kac mechanism. It is VW mirror self dual.*

Proof. Together with W. Nahm we have proven the first part of the proposition in [?, Th. 3.7].

As to the second part, we first have to modify our construction in Sects. 3, 4, since the underlying toroidal model has non-vanishing B-field and non-orthogonal lattice $\Lambda = 1\sqrt{2}\mathcal{D}_4$. However, $\sqrt{2}\mathbb{Z}^4 \subset 1\sqrt{2}\mathcal{D}_4 \subset 1\sqrt{2}\mathbb{Z}^4$. Therefore, we can consider the mirror map induced by fiberwise T-duality on the first two coordinates of the orthogonal torus $\mathbb{R}^4/1\sqrt{2}\mathbb{Z}^4$. Since with respect to the standard basis of $\mathbb{R}^4$

$$B^* = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} :\ \Lambda \otimes \mathbb{R} \longrightarrow \Lambda^* \otimes \mathbb{R}\,, \tag{5.3}$$

this mirror map corresponds to T-duality on a toroidal SCFT in two real dimensions with lattice $1\sqrt{2}\mathcal{D}_2$ and B-field $B^{**} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The latter model again has enhanced symmetry due to the Frenkel-Kac mechanism and is invariant under T-duality. All formulas derived for orthogonal tori with vanishing B-field can therefore

[9] We always omit the origin $0 \in \Delta$ as well.

be applied, and the model is invariant under this version of the VW approach to mirror symmetry on $K3$. □

Remark. *As remarked in* [**?**, Rem. 3.8]*, the bosonic subtheory of* $(\widetilde{2})^4$ *agrees with the bosonic subtheory of the toroidal model on* $\mathbb{R}^4/\mathbb{Z}^4$ *with vanishing B-field. The latter toroidal SCFT is also VW mirror self dual.*

To summarize, the Gepner type model $(\widetilde{2})^4$ allows a discussion of all three versions of mirror symmetry on $K3$ that where mentioned in Sect. 3. It is mirror self dual under all of them.

Now let us compare the induced maps on the fields of the theory:

Proposition 5.4. *On* $(\widetilde{2})^4$*, the mirror maps induced by the VW and GP approaches to mirror symmetry agree.*

Proof. Recall [**?**, Th. 3.7] that $(\widetilde{2})^4$ possesses a left and a right current algebra of type $su(2)_1^6$, such that each field in the theory is entirely specified by its charges under a left and a right $u(1)^6$ current algebra. Moreover, mirror symmetry only acts on the left hand side of each field. Hence it suffices to determine the map induced by mirror symmetry on the left handed current algebra $su(2)_1^6$.

As in the proof of [**?**, Th. 3.7] we denote the $U(1)$ currents in the left handed $su(2)_1^6$ by $J,\ A,\ P,\ Q,\ R,\ S$, and the other two generators of each $su(2)_1$ factor by $J^\pm,\ A^\pm,\ P^\pm,\ Q^\pm,\ R^\pm,\ S^\pm$.

Let us begin with the Gepner type description of $(\widetilde{2})^4$ and the GP approach to mirror symmetry. Let $J_1,\dots,J_4$ denote the $U(1)$ currents of the minimal model factors in $(\widetilde{2})^4$. There are primary fields $\Phi^l_{m,s;\overline{m},\overline{s}}$ with quantum numbers $l\in\{0,1,2\},\ m,\overline{m}\in\mathbb{Z}_8,\ s,\overline{s}\in\mathbb{Z}_4$ in each minimal model factor. Moreover, $X_{i,j}$ denotes the primary field with factor $\Phi^0_{4,2;0,0}$ in the i^{th} and j^{th} component and $\Phi^0_{0,0;0,0}$ elsewhere, and for $Y_{i,j}$ we have $\Phi^0_{-2,2;0,0}$ in the i^{th} and j^{th} factor and $\Phi^0_{2,2;0,0}$ otherwise. The formula between [**?**, (3.18)] and [**?**, (3.19)] gives the $su(2)_1^6$ current algebra in terms of the $J_i,\ X_{i,j},\ Y_{i,j}$. Now note that the GP version of mirror symmetry is induced by $J_i\mapsto -J_i,\ i\in\{1,\dots,4\}$, and therefore $Y_{i,j}\leftrightarrow Y_{k,m}$ with $\{i,j,k,m\}=\{1,\dots,4\}$, whereas the $X_{i,j}$ are invariant. Hence, the induced map on $su(2)_1^6$ is given by

$$
\begin{array}{rlcl}
 & (\mathcal{J},\mathcal{J}^+,\mathcal{J}^-) & \longmapsto & (-\mathcal{J},\mathcal{J}^-,\mathcal{J}^+) \text{ for } \mathcal{J}\in\{J,A,P,Q\},\\
\text{and} & (\mathcal{J},\mathcal{J}^+,\mathcal{J}^-) & \longmapsto & (\mathcal{J},\mathcal{J}^+,\mathcal{J}^-) \text{ for } \mathcal{J}\in\{R,S\}.
\end{array}
\tag{5.4}
$$

In the nonlinear sigma model description of $(\widetilde{2})^4$, the $su(2)_1^6$ current algebra is obtained from $\mathbb{Z}_2$ invariant fields in the underlying toroidal SCFT $\mathcal{C}_T$. There, we have four Majorana fermions $\psi^1,\dots,\psi^4$ and their superpartners, four Abelian $U(1)$ currents $j^1,\dots,j^4$. Let $e_1,\dots,e_4$ denote the standard orthonormal generators of $\mathbb{Z}^4$. As before, we identify $H^1(T,\mathbb{Z})\cong\Lambda^*$, and generators of Λ^* are expressed in terms of the e_i, such that we can interpret the e_i as elements of $H^1(T,\mathbb{R})$. Recall [**?**] that our torus fibration with fiber direction e_1,e_2 is special Lagrangian with respect to the complex structure $\mathcal{I}$ given by $(e_1-ie_3)\wedge(e_2+ie_4)$ (see [**?**]). Accordingly, we define

$$
\psi_\pm^{(1)} := 1\sqrt{2}(\psi^3\pm i\psi^1),\quad \psi_\pm^{(2)} := 1\sqrt{2}(\psi^2\pm i\psi^4).
$$

Hence in comparison to [**?**, (2.1)] we have to rename

$$
(\psi^1,\psi^2,\psi^3,\psi^4)\mapsto(\psi^3,\psi^1,\psi^2,\psi^4),\quad (j^1,j^2,j^3,j^4)\mapsto(j^3,j^1,j^2,j^4).
\tag{5.5}
$$

The $\mathbb{Z}_2$ invariant current algebra $su(2)_1^2$ obtained from the fermions is given in [**?**, (2.2)].

Vertex operators in the toroidal SCFT $\mathcal{C}_T$ are specified by their charges with respect to $(j^1,\dots,j^4)$. Set

$$\forall\, i,j\in\{1,\dots,4\},\ i\neq j:\quad \alpha_{i,j}^{\pm}:=\sqrt{2}(e_i\pm e_j).$$

Then the 24 holomorphic vertex operators with left dimension 1 in $\mathcal{C}_T$ have left handed charges $\alpha_{i,j}^{\pm},\ -\alpha_{i,j}^{\pm}$ with respect to $(j^1,\dots,j^4)$. The $\mathbb{Z}_2$ invariant vertex operator with charges $\alpha_{i,j}^{\pm},\ -\alpha_{i,j}^{\pm}$ is denoted $U_{i,j}^{\pm}$, and $W_{i,j}^{\pm}:=U_{i,j}^{+}\pm U_{i,j}^{-}$ after appropriate normalization. With [**?**, (3.18)] one finds the representation of the missing $su(2)_1^4$ in terms of the $U_{i,j}^{\pm}$. To match with our orbifold conventions, we still have to take (5.5) into account and perform a permutation of the factors of $su(2)_1^4$. Moreover, in the proof of Prop. 5.5 we will need a specific choice of the Cartan torus in each $su(2)_1$ factor: The choice of complex structure $\mathcal{I}$ used in the VW approach to mirror symmetry must match with the choice of $N=2$ superconformal algebra in the $N=4$ superconformal algebra that is obtained from the Gepner construction. This leaves us with a renaming of the $su(2)_1^4$ generators in $\mathcal{C}_T/\mathbb{Z}_2$ by

$$\begin{array}{l}(P,\ Q,\ R,\ S)\mapsto\ (R,\quad i2(S^+-S^-),\quad i2(P^+-P^-),\quad 12(Q^++Q^-));\\ (P^\pm,Q^\pm,R^\pm,S^\pm)\mapsto(R^\pm,S\mp i2(S^++S^-),P\mp i2(P^++P^-),Q\mp 12(Q^+-Q^-))\end{array}$$

with respect to [**?**, (3.18)]. Now we can write out the appropriate expressions for all $\mathcal{J},\ \mathcal{J}^{\pm}$ in (5.4) in terms of fields in the nonlinear sigma model.

From our explanation in the proof of Prop. 5.3, the VW mirror map is induced by $(j^1,j^2,j^3,j^4)\mapsto(-j^1,-j^2,j^3,j^4)$, $(\psi^1,\psi^2,\psi^3,\psi^4)\mapsto(-\psi^1,-\psi^2,\psi^3,\psi^4)$. It therefore leaves $U_{1,2}^{\pm}$ and $U_{3,4}^{\pm}$ invariant and on all other $U_{i,j}^{\pm}$ induces $U_{i,j}^{+}\leftrightarrow U_{i,j}^{-}$. One now checks that the induced map on the $su(2)_1^6$ current algebra in fact agrees with (5.4), proving the assertion. □

Proposition 5.5. *The AMD approach to mirror symmetry on $\widetilde{(2)}^4$ is compatible with the GP$\overset{Prop.\ 5.4}{=}$VW approaches.*

Proof. To study the AMD approach to mirror symmetry on $\widetilde{(2)}^4$ we invoke the toric description given in Prop. 5.2. In the following, we will use the notations introduced in Props. 5.2 and 5.4. Let us denote the four vertices of Δ in the order they are listed in (5.2) by $A,\ B,\ C,\ D$, and the center of the edge between A and B by AB etc. The origin $0\in\Delta$ is denoted o. The corresponding elements of $Pic_{tor}(\Delta)$ are $\omega_A,\ \omega_B,\dots;\omega_{AB},\ \omega_{AC},\dots;\omega_o$. By the proof of Prop. 5.2

$$\begin{array}{rcl}\omega_A^2 = \omega_B^2 = \omega_C^2 = \omega_D^2 &=& -2,\\ \omega_{AB}^2 = \omega_{AC}^2=\omega_{AD}^2=\omega_{BC}^2=\omega_{BD}^2=\omega_{CD}^2 &=& -4,\\ \omega_o^2 = (\omega_A+\cdots+\omega_D+\omega_{AB}+\cdots+\omega_{CD})^2 &=& 16.\end{array}$$

On inspection of the relations between the various cocycles $\omega_\bullet$ one finds

$$\omega_o=-2(-2\omega_A+2\omega_B+2\omega_C-\omega_{AD}+2\omega_{BC}+\omega_{BD}+\omega_{CD})=2\omega_{\tilde o}$$

with $\omega_{\tilde{o}} \in Pic_{tor}(\Delta^*)$. Moreover,

$$\begin{aligned}
\omega_A &= -12(\omega_{\tilde{o}} + \omega_{AB} + \omega_{AC} + \omega_{AD}),\\
\omega_B &= -12(\omega_{\tilde{o}} + \omega_{AB} + \omega_{BC} + \omega_{BD}),\\
\omega_C &= -12(\omega_{\tilde{o}} + \omega_{CD} + \omega_{AC} + \omega_{BC}),\\
\omega_D &= -12(\omega_{\tilde{o}} + \omega_{CD} + \omega_{AD} + \omega_{BD}).
\end{aligned} \tag{5.6}$$

The AMD approach to mirror symmetry in its improved version by Rohsiepe (see Prop. 3.1) implies that the mirror map induces a lattice automorphism which acts on

$$Pic_{tor}(\Delta) \perp Pic(\Delta^*) \subset \Gamma^{2,18} \subset H^2(X, \mathbb{Z})$$

such that $Pic_{tor}(\Delta)$ is mapped onto the sublattice $Pic_{tor}(\Delta^*) \subset Pic(\Delta^*)$ of rank 7.

To compare with the GP approach to mirror symmetry note that the mirror of $(2)^4$ corresponds to a family of Calabi-Yau toric hypersurfaces with generic Picard lattice of rank 1. In $(2)^4$ we therefore have $20 - 1 = 19$ states with conformal dimensions $(1/4, 1/4)$ which are uncharged under the $U(1)$ current of the $N = 2$ superconformal algebra specified by the Gepner construction and which also belong to $(2)^{\otimes 4}$. As we shall explain below, by the correspondence between Gepner states and complex structure deformations for the mirror in terms of monomials [**?**, **?**] together with the monomial-divisor map [**?**], these 19 states correspond to generators of the QUANTUM PICARD LATTICE $Pic \oplus U$ (see [**?**]) associated to $(2)^4$. One of them generates deformations of the volume of the relevant toric Calabi-Yau hypersurface and does not correspond to a vector in the ordinary Picard lattice Pic. To generate the latter, an appropriate element of the Kähler cone has to be determined separately.

The family corresponding to $(\widetilde{2})^4$ is obtained from that corresponding to $(2)^4$ by a $\mathbb{Z}_2^2$ orbifold procedure and has generic Picard lattice $Pic(\Delta)$ of rank 13 as in (5.1). Among the 19 states mentioned before, seven are invariant under the $\mathbb{Z}_2^2$ action and therefore should correspond to elements of $Pic_{tor}(\Delta) \oplus U = \check{Pic}(\Delta^*) \oplus U$ above: Six of type $\Xi_{i,j}$ with $\Phi^0_{1,1;1,1}$ in the i^{th} and j^{th} factors and $\Phi^0_{-1,-1;-1,-1}$ elsewhere, and $(\Phi^1_{2,1;2,1})^{\otimes 4}$. The geometric counterpart of each of these Gepner states in the family described by Δ is determined by the Abelian symmetries of $(\widetilde{2})^4$. More precisely, we can apply the operator $(\Phi^0_{-1,-1;-1,-1})^{\otimes 4}$ of spectral flow to obtain states in the (c, c) ring of the Gepner type model and compare with deformations of the complex structure for members of the mirror family. By the monomial-divisor map [**?**] they correspond to specific elements of $Pic_{tor}(\Delta) \oplus U$, and if these elements belong to $Pic_{tor}(\Delta)$ they can be labeled by lattice vectors in $\partial\Delta$. Here we have $\widetilde{\Xi}_{i,j}$ with $\Phi^0_{-2,-2;-2,-2} \sim \Phi^2_{2,0;2,0}$ in the i^{th} and j^{th} factors and $\Phi^0_{0,0;0,0}$ elsewhere which by [**?**, **?**] corresponds to the monomial $x_i^2 x_j^2 \in \mathbb{C}[x_A, x_B, x_C, x_D]$, or, in terms of Δ, the center of an edge. The remaining state is mapped to $(\Phi^1_{1,0;1,0})^{\otimes 4}$ under spectral flow and corresponds to the generator of volume deformations of the toric Calabi-Yau hypersurface which will not be discussed in the following.

Recall from the proof of Prop. 5.2 that for a choice of generators of $Pic_{tor}(\Delta)$ at most one vertex of Δ can be omitted such that the six lattice vectors corresponding to the $\Xi_{i,j}$ cannot be expected to generate a primitive sublattice of $Pic_{tor}(\Delta)$. However, using the exact field-to-field identifications of [**?**, Th. 3.7] we can explicitly determine the geometric counterparts of these states in our nonlinear sigma model interpretation of $\widetilde{(2)}^4$, which will enable us to calculate intersection numbers and perform a compatibility check between the AMD and the GP$\stackrel{\text{Prop. 5.4}}{=}$VW approaches to mirror symmetry. Here, $(t_1, t_2, t_3, t_4) \in \mathbb{F}_2^4$ labels the fixed point $12\sum_j t_j\lambda_j$ with appropriate generators λ_j of the torus lattice $\Lambda = 1\sqrt{2}\mathcal{D}_4$ with $T = \mathbb{R}^4/\Lambda$:

$$\lambda_1 := 1\sqrt{2}(e_1 + e_3),\ \lambda_2 := 1\sqrt{2}(e_1 - e_3),\ \lambda_3 := 1\sqrt{2}(e_2 + e_3),\ \lambda_4 := 1\sqrt{2}(e_4 - e_1).$$

As before, by $e_1, \dots, e_4$ we denote the standard orthogonal generators of $\mathbb{Z}^4 \subset \mathbb{R}^4$, where $\mathbb{R}^4$ is identified with its dual by the use of the standard scalar product. By $\mu_1, \dots, \mu_4$ we denote the dual basis with respect to $\lambda_1, \dots, \lambda_4$ which is readily interpreted as basis of $H^1(T, \mathbb{Z})$. Then, four of the above mentioned six states $\Xi_{i,j}$ are identified with linear combinations of twist fields $T_t,\ t \in \mathbb{F}_2^4$, in the nonlinear sigma model description of $\widetilde{(2)}^4$. From the proof of [**?**, Th. 3.7] together with the renaming that was observed in the proof of Prop. 5.4 we find the explicit linear combinations (see the formula below). From Prop. 4.2 we know that $T_t,\ t \in \mathbb{F}_2^4$, corresponds to a cocycle $(-E_t) \in H^{even}(X, \mathbb{R})$ of the relevant Kummer surface X. With (3.1) it follows that $\widehat{E}_t = E_t + 12\widehat{\upsilon} \in H^2(X, \mathbb{Z})$, and all in all we find that the following identifications can be made in $H^2(X, \mathbb{Z})$:

$$\begin{aligned}
\Xi_{1,3} \ \widehat{=}\ \omega_{AC} &= 12\sum_{j,k\in\mathbb{F}_2}(-1)^{j+k}\widehat{E}_{(j,j,k,k)} + 12\sum_{j,k\in\mathbb{F}_2}(-1)^{j+k}\widehat{E}_{(j,j,k,k+1)},\\
\Xi_{2,4} \ \widehat{=}\ \omega_{BD} &= 12\sum_{j,k\in\mathbb{F}_2}(-1)^{j+k}\widehat{E}_{(j,j,k,k)} - 12\sum_{j,k\in\mathbb{F}_2}(-1)^{j+k}\widehat{E}_{(j,j,k,k+1)},\\
\Xi_{1,4} \ \widehat{=}\ \omega_{AD} &= 12\sum_{j,k\in\mathbb{F}_2}(-1)^{j}\widehat{E}_{(j,j,k,k)} + 12\sum_{j,k\in\mathbb{F}_2}(-1)^{j}\widehat{E}_{(j,j,k,k+1)},\\
\Xi_{2,3} \ \widehat{=}\ \omega_{BC} &= 12\sum_{j,k\in\mathbb{F}_2}(-1)^{j}\widehat{E}_{(j,j,k,k)} - 12\sum_{j,k\in\mathbb{F}_2}(-1)^{j}\widehat{E}_{(j,j,k,k+1)},\\
\Xi_{1,2} \ \widehat{=}\ \omega_{AB} &= \sqrt{2}\,(\ e_1\wedge(e_3-e_2)\ +\ (e_2+e_3)\wedge e_4\ )\\
&= \sqrt{2}\,(\ (\mu_1-\mu_2)\wedge\mu_4\ -\ \mu_1\wedge\mu_2\ +\ \mu_3\wedge\mu_4\ ),\\
\Xi_{3,4} \ \widehat{=}\ \omega_{CD} &= \sqrt{2}\,(\ e_1\wedge(e_2+e_3)\ +\ (e_2-e_3)\wedge e_4\ )\\
&= \sqrt{2}\,(\ (\mu_1+\mu_2)\wedge\mu_3\ -\ \mu_1\wedge\mu_2\ +\ \mu_3\wedge\mu_4\ ),\\
\omega_{\widetilde{o}} &= \sqrt{2}\,(\ e_1\wedge(e_2-e_3)\ +\ (e_2+e_3)\wedge e_4\ )\\
&= \sqrt{2}\,(\ \mu_1\wedge\mu_2\ +\ \mu_3\wedge\mu_4\ ).
\end{aligned}$$

One first checks that the GP=VW approach to mirror symmetry indeed maps any rank 7 lattice $P \subset H^{even}(X, \mathbb{Z})$ containing these seven vectors onto a lattice in $P^\perp$. Moreover, all of them are pairwise orthogonal primitive lattice vectors, and $\omega_{\widetilde{o}}^2 = 4$, whereas for the other $\omega_\bullet$ listed above we have $\omega_\bullet^2 = -4$, in accordance with the toric picture. To show that the primitive sublattice of $H^2(X, \mathbb{Z})$ containing these vectors indeed is $\langle 4\rangle \oplus \langle -2\rangle^6 \cong Pic_{tor}(\Delta)$, we need to show that the corresponding

$\omega_A, \ldots, \omega_D$ given in (5.6) are lattice vectors. Indeed,

$$\begin{aligned}
\omega_A &= -\sqrt{2}\mu_3 \wedge \mu_4 \\
&\quad -1\sqrt{2}(\mu_1 - \mu_2) \wedge \mu_4 - 12\left(\widehat{E}_{(0,0,0,0)} - \widehat{E}_{(1,1,0,0)} + \widehat{E}_{(0,0,0,1)} - \widehat{E}_{(1,1,0,1)}\right), \\
\omega_B &= -\sqrt{2}\mu_3 \wedge \mu_4 \\
&\quad -1\sqrt{2}(\mu_1 - \mu_2) \wedge \mu_4 - 12\left(\widehat{E}_{(0,0,0,0)} - \widehat{E}_{(1,1,0,0)} - \widehat{E}_{(0,0,0,1)} + \widehat{E}_{(1,1,0,1)}\right), \\
\omega_C &= -\sqrt{2}\mu_3 \wedge \mu_4 \\
&\quad -1\sqrt{2}(\mu_1 + \mu_2) \wedge \mu_3 - 12\left(\widehat{E}_{(0,0,0,0)} - \widehat{E}_{(1,1,0,0)} - \widehat{E}_{(0,0,1,0)} + \widehat{E}_{(1,1,1,0)}\right), \\
\omega_D &= -\sqrt{2}\mu_3 \wedge \mu_4 \\
&\quad -1\sqrt{2}(\mu_1 + \mu_2) \wedge \mu_3 - 12\left(\widehat{E}_{(0,0,0,0)} - \widehat{E}_{(1,1,0,0)} + \widehat{E}_{(0,0,1,0)} - \widehat{E}_{(1,1,1,0)}\right)
\end{aligned}$$

are of the type listed in (2.2). □

6. Conclusions

In this note, we have given a brief description of the known structure of the moduli space $\mathcal{M}$ of $N = (4,4)$ SCFTs with $c = 6$ and have spelled out some of its explicit connections to geometry.

The algebraic structure of $\mathcal{M}$ had been known before [**?**, **?**, **?**, **?**], but only recently [**?**, **?**] the location of orbifold CFTs on $K3$ that are obtained from toroidal theories in $\mathcal{M}$ was described in terms of geometric quantities. The main problem is the determination of the B-field values in a geometric interpretation of such an orbifold CFT on the corresponding orbifold limit X of $K3$. The B-field is nonzero in direction of each component of the exceptional divisor in X [**?**] and can be determined explicitly by a generalization of Nikulin's methods to describe the Kummer lattice and its embedding in $H^2(X, \mathbb{Z})$. We argue that these nonzero B-field values can be understood as artifact from the specific choice of geometric interpretation on the orbifold limit X. Moreover, they have a straightforward explanation in terms of the classical McKay correspondence which we venture to conjecture should allow a determination of the B-field values in a more general setting, too. This also provides an explicit geometric understanding of B-fields, at least in the context of SCFTs on $K3$.

The second part of this note is devoted to a discussion of mirror symmetry on the $\mathbb{Z}_N$ orbifold CFTs discussed before. We investigate a version of mirror symmetry on elliptically fibred $K3$ surfaces that is induced by fiberwise T-duality on nonsingular fibers. This part of the note is a summary of [**?**]. Our explicit knowledge of the relevant lattices allows the determination of the corresponding automorphism of the lattice of integral cohomology on $K3$. On the other hand, the action on the relevant states of our SCFTs is found to have the structure of a fiberwise discrete $\mathbb{Z}_N$ type Fourier transform. A comparison of the geometric and the conformal field theoretic mirror maps now provides us with a dictionary to directly translate geometric data into conformal field theoretic ones. The "quantum" $\mathbb{Z}_N$ symmetry of the twisted sector of $\mathbb{Z}_N$ orbifold CFTs is confirmed to have a geometric meaning. Moreover, the action of the toroidal vertex operator algebra on twisted ground states of the orbifold CFT translates into a natural action on line bundles on $K3$ which are obtained by the classical McKay correspondence from $\mathbb{Z}_N$ equivariant line bundles on the underlying torus. In fact, our dictionary can be interpreted as CFT version of Ito/Reid's "dual" McKay correspondence [?]. It bears the additional property of being compatible with the Weyl algebra representation of the toroidal vertex operator algebra on twisted ground states and $\mathbb{Z}_N$ equivariant flat line bundles, respectively. This is also the explicit map that has to be used in order to resolve the objection of [?] to Ruan's conjecture [?] on the orbifold cohomology of hyperkähler surfaces[10] (see [?]).

Apart from our approach to mirror symmetry, which is based on ideas by Vafa and Witten [?], the literature contains many statements about mirror symmetry on $K3$ which at first sight appear not to be compatible. Therefore, in this note we have included the discussion of an example which allows for a comparison of our approach with two other mainstream versions of mirror symmetry, due to Greene and Plesser [?] on the one hand and Aspinwall/Morrison and Dolgachev [?, ?] on the other. In fact, with an emendation of the latter approach in a toric setting [?] due to Rohsiepe [?, ?] we can show that all three versions of mirror symmetry on $K3$ agree for our example, at least to the extent of comparability.

The virtue of our particular example is the fact that it is mirror self dual. There are other examples of SCFTs in $\mathcal{M}$ which allow the application of all three versions of mirror symmetry but are not as well behaved with respect to a comparison. E.g., the Gepner model $(2)^4$ has a nonlinear sigma model interpretation as the $\mathbb{Z}_4$ orbifold CFT of the toroidal model on $\mathbb{R}^4/\mathbb{Z}^4$ with vanishing B-field [?, Th. 3.5]. This model appears to be mirror self dual under our approach to mirror symmetry since the underlying toroidal theory is, but it is not mirror self dual under the other two versions mentioned above. The resolution to this puzzle probably again is the special role of the B-field for orbifold CFTs on $K3$: It is known that a shift of the B-field $B \in H^2(X, \mathbb{R})$ in one of our SCFTs by an integral two form does not change the physics in our theory. Therefore, in [?, ?, ?] we have effectively considered the B-field as element of $H^2(X, \mathbb{R})/H^2(X, \mathbb{Z})$. In particular, we do not keep track of integral B-field shifts which might occur under mirror symmetry on an orbifold CFT of a toroidal theory $\mathcal{C}_T/\mathbb{Z}_N$, even if $\mathcal{C}_T$ is mirror self dual. Since we have given a geometric interpretation of the role of the B-field by using the classical and the dual McKay correspondence, it might be interesting to study this phenomenon in greater detail.

Summarizing, we hope to have convinced the reader that orbifold CFTs on $K3$ are simple enough to make explicit mathematical statements and complicated enough to provide a rich playground for a study of interrelations between geometry and conformal field theory.

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[10] The fact that our transformation resolves this objection was explained to us by Yongbin Ruan [?] and goes back to an earlier observation by Edward Witten.